

# On $l^p$ -multipliers of functions analytic in the disk <sup>1</sup>

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We consider bounded analytic functions in domains generated by sets that have Littlewood–Paley property. We show that each such function is an  $l^p$ -multiplier.

References: 12 items.

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Given a function  $f$  analytic in the unit disk  $D = \{z \in \mathbb{C} : |z| < 1\}$  of the complex plane  $\mathbb{C}$ , consider its Taylor expansion:

$$f(z) = \sum_{n \geq 0} \widehat{f}(n) z^n, \quad z \in D. \quad (1)$$

For  $1 \leq p \leq \infty$  let  $A_p^+(D)$  denote the space of all functions (1) such that the sequence of Taylor coefficients  $\widehat{f} = \{\widehat{f}(n), n = 0, 1, \dots\}$  belongs to  $l^p$ . For  $f \in A_p^+(D)$  we put  $\|f\|_{A_p^+(D)} = \|\widehat{f}\|_{l^p}$ . A function  $m$  analytic in  $D$  is called an  $l^p$ -multiplier if for every function  $f$  in  $A_p^+(D)$  we have  $m \cdot f \in A_p^+(D)$ . We denote the class of all these multipliers by  $M_p^+(D)$ . This class is a Banach algebra with respect to the natural norm

$$\|m\|_{M_p^+(D)} = \sup_{\|f\|_{A_p^+(D)} \leq 1} \|m \cdot f\|_{A_p^+(D)}$$

and the usual multiplication of functions. The classes  $M_p^+(D)$  were studied in [1]–[6]. <sup>2</sup> We note that the case when  $p \neq 1, \infty, 2$  is of a special interest. It is well known that  $M_p^+(D) = M_q^+(D)$  if  $1/p + 1/q = 1$ , and

$$A_1^+(D) = M_1^+(D) = M_\infty^+(D) \subseteq M_p^+(D) \subseteq M_2^+(D) = H^\infty(D),$$

where  $H^\infty(D)$  is the Hardy space of bounded analytic functions in  $D$ .

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<sup>2</sup>There is a minor inconsistency in § 6 of the author's work [6]. Instead of the written “the Poisson integral” there should be “the Riesz projection  $\sum_{k=-\infty}^{\infty} c_k e^{ikt} \rightarrow \sum_{k \geq 0} c_k z^k$ ”.

Let  $\Omega \subseteq \mathbb{C}$  be an open domain which contains the disk  $D$ . We shall present a class of domains  $\Omega$  such that each bounded analytic function in  $\Omega$  belongs to  $M_p^+(D)$ . The case when  $\Omega$  contains the closure of  $D$  is trivial; in this case each bounded analytic function in  $\Omega$  belongs to  $A_1^+(D)$  and hence belongs to  $M_p^+(D)$  for all  $p$ ,  $1 \leq p \leq \infty$ . The nontrivial case is the case when the boundary of  $\Omega$  has common points with the boundary  $\partial D = \{z \in \mathbb{C} : |z| = 1\}$  of the disk  $D$ .

It was shown by Vinogradov [2] that if  $r > 1$ ,  $0 \leq \alpha < \pi/2$ , and  $m$  is a bounded analytic function in the domain

$$\Omega_0 = \{z \in \mathbb{C} : |z| < r, \alpha < \arg(z - 1) < 2\pi - \alpha\}, \quad (2)$$

then  $m \in \bigcap_{1 < p < \infty} M_p^+(D)$ . Using this result Vinogradov gave the first examples of nontrivial (i.e., infinite) Blaschke products in  $M_p^+(D)$ . Note that the boundary of a domain (2) has only one common point with the boundary of  $D$  (namely the point  $z = 1$ ). As we shall see, a statement similar to Vinogradov's result holds for domains of a much wider class. Functions analytic in the domains considered below can have uncountable set of singularities.<sup>3</sup>

As is usual, for an arbitrary domain  $\Omega \subseteq \mathbb{C}$  by  $H^\infty(\Omega)$  we denote the Hardy space of all bounded analytic functions in  $\Omega$ . For  $g \in H^\infty(\Omega)$  we put  $\|g\|_{H^\infty(\Omega)} = \sup_{z \in \Omega} |g(z)|$ .

Let  $J$  be an arc in the boundary circle  $\partial D$ . Assume that the length  $|J|$  of  $J$  is strictly less than  $\pi$ . Let  $T_J$  be an arbitrary open isosceles triangle, whose base is the chord that spans the arc  $J$ , and whose sides lie outside of  $D$ . Denote by  $\theta_{T_J}$  the angle between  $\partial D$  and a side of  $T_J$ .

Consider an arbitrary closed set  $F \subseteq \partial D$ . Let  $\tau(F)$  be the family of all arcs complimentary to  $F$  (i.e., of all connected components of the compliment  $\partial D \setminus F$ ). We assume that each arc of the family  $\tau(F)$  has length strictly less than  $\pi$ . Consider the domain

$$\Omega_F = D \cup \bigcup_{J \in \tau(F)} T_J,$$

and require in addition that

$$\inf_{J \in \tau(F)} \theta_{T_J} > 0. \quad (3)$$

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<sup>3</sup>We note by the way that the condition  $\alpha < \pi/2$  in the Vinogradov theorem is essential. For example, the function  $S(z) = \exp\{(z+1)/(z-1)\}$  is bounded in the half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ , but as it was shown by Verbitskii [4]  $S$  belongs to  $M_p^+(D)$  only in the trivial case  $p = 2$ .

We call a domain  $\Omega_F$ , obtained in this way, a star-like domain generated by  $F$ .

We shall show that under a certain condition imposed on a set  $F \subseteq \partial D$ , every function, bounded and analytic in  $\Omega_F$ , belongs to  $M_p^+(D)$ .

Let  $E$  be a closed set of Lebesgue measure zero in the line  $\mathbb{R}$ . Consider the family  $\tau(E)$  of all intervals complimentary to  $E$  (i.e., of all connected components of the compliment  $\mathbb{R} \setminus E$ ). For an arbitrary interval  $I \subseteq \mathbb{R}$  define the operator  $S_I$  by

$$\widehat{S_I(f)} = 1_I \widehat{f}, \quad f \in L^p \cap L^2(\mathbb{R}),$$

where  $\widehat{\phantom{x}}$  is the Fourier transform and  $1_I$  is the characteristic function of  $I$  (i.e.,  $1_I(t) = 1$  for  $t \in I$ ,  $1_I(t) = 0$  for  $t \notin I$ ). Following [7], we say that a set  $E$  has property LP( $p$ ) ( $1 < p < \infty$ ) if the corresponding Littlewood–Paley quadratic function

$$S(f) = \left( \sum_{I \in \tau(E)} |S_I(f)|^2 \right)^{1/2}$$

satisfies

$$c_1(p) \|f\|_{L^p(\mathbb{R})} \leq \|S(f)\|_{L^p(\mathbb{R})} \leq c_2(p) \|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R})$$

(with positive constants  $c_1(p)$ ,  $c_2(p)$  independent of  $f$ ). In the case when  $E$  has property LP( $p$ ) for all  $p$ ,  $1 < p < \infty$ , we say that  $E$  has property LP.

Let now  $F$  be a closed set of measure zero in the boundary circle  $\partial D$ . We say that  $F$  has property LP( $p$ ) or property LP if  $F = \{e^{it}, t \in E\}$ , where  $E \subseteq [0, 2\pi]$  is a set that has property LP( $p$ ) or property LP, respectively.

*Remark 1.* A classical example of an infinite set  $E \subseteq \mathbb{R}$  that has property LP is  $E = \{\pm 2^k, k \in \mathbb{Z}\} \cup \{0\}$ , where  $\mathbb{Z}$  is the set of integers. At the same time there exist uncountable sets that have property LP. This was first established by Hare and Klemes [8]. The existence of such sets was also noted in [9], see details in [10, § 4]. Let us state the corresponding result for sets in  $\partial D$ . For each  $p$ ,  $1 < p < \infty$ , there is a constant  $\beta_p$  ( $0 < \beta_p < 1$ ) such that the following holds. Let  $F \subseteq \partial D$  be a closed set of measure zero. Suppose that the arcs  $J_k$ ,  $k = 1, 2, \dots$ , complimentary to  $F$ , being enumerated so that their lengths do not increase, satisfy  $|J_{k+1}|/|J_k| \leq \beta_p$  for all sufficiently large  $k$ . Then  $F$  has property LP( $p$ ). This in turn implies that if  $\lim_{k \rightarrow \infty} |J_{k+1}|/|J_k| = 0$ , then  $F$  has property LP.

The result of this note is the following theorem.

**Theorem.** *Suppose that a set  $F \subseteq \partial D$  has property LP( $p$ ), and  $\Omega_F$  is a star-like domain generated by  $F$ . Then  $H^\infty(\Omega_F) \subseteq M_p^+(D)$ . If  $F$  has property LP, then  $H^\infty(\Omega_F) \subseteq \bigcap_{1 < p < \infty} M_p^+(D)$ .*

*Proof.* Let  $G$  be an Abelian group and let  $\Gamma$  be the group dual to  $G$ . Consider a function  $m \in L^\infty(\Gamma)$  and the operator  $Q$  defined by

$$\widehat{Qf} = m\widehat{f}, \quad f \in L^p \cap L^2(G),$$

where  $\widehat{\phantom{x}}$  stands for the Fourier transform on  $G$ . The function  $m$  is called an  $L^p$ -Fourier multiplier if the corresponding operator  $Q$  is a bounded operator from  $L^p(G)$  to itself ( $1 \leq p \leq \infty$ ). Denote the class of all these multipliers by  $M_p(\Gamma)$  and put  $\|m\|_{M_p(\Gamma)} = \|Q\|_{L^p(G) \rightarrow L^p(G)}$ . The relation between the multipliers on the line  $\mathbb{R}$  and on the circle  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  is well known [11] (see also [12]). We shall need the Jodeit theorem [12] on the periodic extension of multipliers. According to this theorem, if  $f \in M_p(\mathbb{R})$  is a function that vanishes outside of the interval  $[0, 2\pi]$  and  $g$  is the  $2\pi$ -periodic function that coincides with  $f$  on  $[0, 2\pi]$ , then  $g \in M_p(\mathbb{T})$ . Note that there is a direct relation between the spaces  $M_p^+(D)$  and  $M_p(\mathbb{T})$ . Given a function  $m \in H^\infty(D)$  consider its (non-tangential) boundary function  $m^*(t) = m(e^{it})$ . The conditions  $m \in M_p^+(D)$  and  $m^* \in M_p(\mathbb{T})$  are equivalent [3] (see also [5]).

We shall also need the following statement. Let  $E \subseteq \mathbb{R}$  be a set that has property LP( $p$ ). Suppose that a function  $f \in L^\infty(\mathbb{R})$  is continuously differentiable on each interval complimentary to  $E$ , and its derivative  $f'$  satisfies

$$|f'(t)| \leq \frac{c}{\text{dist}(t, E)}, \quad t \in \mathbb{R} \setminus E, \quad (4)$$

where  $\text{dist}(t, E)$  stands for the distance from a point  $t$  to the set  $E$  and  $c > 0$  does not depend on  $t$ . Then  $f \in M_p(\mathbb{R})$ . This result of Sjögren and Sjölin [7] generalizes the well known Mikhlin–Hörmander theorem.

We note now that condition (3) implies the existence of a positive constant  $c = c(\Omega_F)$  such that if  $e^{it} \in \partial D \setminus F$ , then the circle centered at  $e^{it}$  and of radius  $r(t) = c \cdot \text{dist}(e^{it}, F)$  lies in  $\Omega_F$ . Denote this circle by  $\gamma(t)$ . Let  $m \in H^\infty(\Omega_F)$ . Consider an arc  $J$  complimentary to  $F$ . Let  $e^{it} \in J$ . Consider

the corresponding circle  $\gamma(t)$ . For an arbitrary point  $z$  that lies inside  $\gamma(t)$  we have

$$m'(z) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - z)^2} d\zeta.$$

In particular,

$$m'(e^{it}) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - e^{it})^2} d\zeta.$$

Hence, for the derivative  $(m^*)'$  of the boundary function  $m^*(t) = m(e^{it})$  we obtain

$$\begin{aligned} |(m^*)'(t)| &= |ie^{it}m'(e^{it})| = \left| \frac{1}{2\pi i} \int_{\gamma(t)} \frac{m(\zeta)}{(\zeta - e^{it})^2} d\zeta \right| \leq \\ &\leq \frac{1}{2\pi} \int_{\gamma(t)} \frac{|m(\zeta)|}{|\zeta - e^{it}|^2} |d\zeta| \leq \frac{1}{2\pi} 2\pi r(t) \|m\|_{H^\infty(\Omega_F)} \frac{1}{(r(t))^2} = \\ &= c_1(\Omega_F) \|m\|_{H^\infty(\Omega_F)} \frac{1}{\text{dist}(e^{it}, F)}. \end{aligned} \quad (5)$$

Let  $E \subseteq [0, 2\pi]$  be a set such that  $F = \{e^{it}, t \in E\}$  and  $E$  has property  $\text{LP}(p)$ . Without loss of generality we can assume that  $E$  contains the points 0 and  $2\pi$ . Define a function  $f$  on  $\mathbb{R}$  by  $f(t) = 1_{[0, 2\pi]}(t)m^*(t)$ ,  $t \in \mathbb{R}$ . We see that (see (5)) the function  $f$  satisfies (4). Therefore, by Sjögren–Sjölin theorem, we have  $f \in M_p(\mathbb{R})$ . Hence, using the Jodeit theorem, we obtain  $m^* \in M_p(\mathbb{T})$ . Taking into account the relation between multipliers on  $\mathbb{T}$  and multipliers of functions analytic in the disk  $D$ , we obtain  $m \in M_p^+(D)$ .

*Remark 2.* As far as the author knows, the question on the existence of a set that has property  $\text{LP}(p)$  for some  $p$ ,  $p \neq 2$ , but does not have property  $\text{LP}$  is open.

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